Practical Network Coding on Three-Node Point-to-Point Relay Networks

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***Abstract*—We study the three-node point-to-point relay network which consists of two terminal nodes and one relay node between them and investigate practical network coding schemes for the network. The rate region achievable by the practical network coding schemes is obtained, and a procedure for constructing the linear network codes that achieve the rate pairs in the achievable rate region is presented. In addition, we show that the use of network coding rather than routing alone enlarges the achievable rate region, in particular increases the maximum equal-rate throughput.**

I. Introduction

In a traditional cellular architecture where the base stations communicate with the mobile users directly, the data rates of the users at the cell edge are severely limited due to strong interference from neighboring cells and large propagation loss from the serving base station. One way to improve the throughput for these cell-edge users is by placing relay nodes at different locations in the cell. Since the relay nodes typically have higher and stronger antennas than the mobile users, they have better channels than the mobile users to the base station. In addition, the mobile users that are close to some relay nodes can receive signals from the relay nodes with high signal to interference plus noise ratio (SINR) due to close proximity. Therefore, the installation of relay nodes generally improves the overall throughput of the system [1]–[3].

For a given mobile user and its associated base station, the introduction of a relay node virtually creates a *two-way relay channel* (TRC) [4], in which the two terminals corresponding to the mobile user and the base station exchange messages with the help of the relay between them. A number of achievable rate regions for the TRC have been obtained in [4]–[11]. However, the transmission schemes proposed in all the above studies are far from practical due to their high complexity. Therefore, we focus on investigating practical coding schemes for the TRC. Since it is difficult to sufficiently isolate a received wireless signal from a simultaneously transmitted wireless signal in the same frequency spectrum, we assume that all the nodes in the TRC are half-duplex, which means they cannot transmit and receive information at the same time.

Network coding, first studied by Ahlswede *et al.* [12], reveals that if coding is applied at the nodes in a network rather than routing alone, the network capacity can be increased.

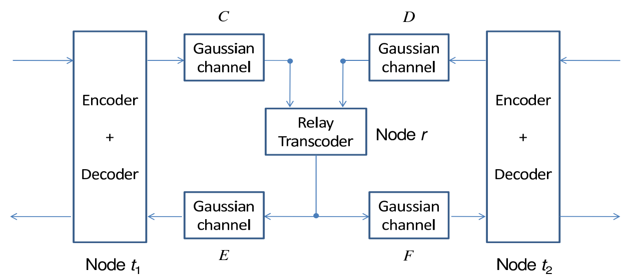


Fig. 1. A half-duplex TRC modeled as a network of point-to-point Gaussian channels.

In addition, practical capacity-approaching coding schemes for a point-to-point bandlimited Gaussian channel such as Turbo codes [13] and LDPC codes [14] are well understood. Therefore, we study practical network coding schemes on the TRC by modeling the TRC as a *three-node point-to-point relay network* consisting of four independent point-to-point bandlimited Gaussian channels as shown in Figure 1, where the two terminals are denoted by t_1 and t_2 and the relay is denoted by r.

This paper is organized as follows. Section II presents the formulation of the three-node point-to-point relay network. In Section III, practical network coding schemes for the network are investigated and the rate region achievable by the practical network coding schemes is obtained. In addition, a procedure for constructing the linear network codes that achieve the rate pairs in the achievable rate region is presented. Section IV shows that the use of network coding rather than routing alone enlarges the achievable rate region, in particular increases the maximum equal-rate throughput. Section V concludes this paper.

II. Three-node point-to-point relay network

Let (u, v) denote the bandlimited Gaussian channel from u to v in the three-node point-to-point relay network where u and v are two adjacent nodes. For comprehensive information-theoretic treatments of Gaussian channels, we refer the reader to [15]–[17]. Let C, D, E and F denote the channel capacities of (t_1, r), (r, t_2), (t_2, r) and (r, t_1) respectively. This is illustrated in Figure 1. It is well known from the channel coding theorem that an information rate can be achieved asymptotically by some transmission scheme with arbitrarily small probability of error if and only if it is less than or equal to the channel capacity. Therefore, for any transmission link (u, v), u can transmit at rate R reliably to v if and only if R is less than the channel capacity of (u, v).

Without loss of generality, we assume in the rest of this paper that the noise power of channel (r, t_1) is less than the noise power of channel (r, t_2) when r transmits, which implies that

(i) $D \leq F$ (1)

(ii) any transmission scheme that transmits information reliably from r to node t_2 can also transmit information reliably from r to node t_1.

Note that channels (r, t_1) and (r, t_2) share a common input (cf. Figure 1). It then follows that r can broadcast information reliably from r to both node t_1 and node t_2 at rate R if and only if $R \leq \min\{D, F\} = D$. We call D the *broadcast capacity*.

In this paper, we study practical network coding schemes that consist of the following four types of transmissions:

Type 1: Node t_1 transmits messages reliably to r at rate C.

Type 2: Node t_2 transmits messages reliably to r at rate E.

Type 3: Node r broadcasts messages reliably to both node t_1 and node t_2 at rate D, the broadcast capacity.

Type 4: Node r transmits messages reliably to node t_1 at rate F.

Without loss of generality, we assume that the transmissions in the network consist of four sessions such that the transmissions in the i^{th} session are of Type i for $i = 1, 2, 3, 4$. We call the transmission scheme described above a *four-session network coding scheme* or simply a *four-session scheme*.

Note that in the subsequent computation of the achievable rate region, only the fractions of time allocated to the four types of transmission matter. Thus the four-session scheme is the most general scheme if these four types of transmission are considered.

III. ACHIEVABLE RATE REGION $\mathcal{T}_4$

Definition 1: A vector $[\, p_1\ p_2\ p_3\ p_4 \,]$ is called a four-session allocation if p_1, p_2, p_3 and p_4 are non-negative real numbers such that

$$p_1 + p_2 + p_3 + p_4 \leq 1. \tag{2}$$

Definition 2: Let τ be a four-session scheme on the three-node point-to-point relay network and $s = [\, p_1\ p_2\ p_3\ p_4 \,]$ be a four-session allocation. The transmission scheme τ is called a transmission scheme with configuration s if under the scheme, a fraction p_i of the time is allocated to Type i transmissions for $i = 1, 2, 3, 4$.

Definition 3: Let s be a four-session allocation, and R_1 and R_2 be two non-negative real numbers. An information rate pair (R_1, R_2) is achievable by a four-session scheme τ if under the scheme, node t_1 and node t_2 exchange independent messages such that node t_1 can transmit messages reliably to node t_2 at an average rate higher than or equal to R_1 and node t_2 can transmit messages reliably to node t_1 at an average rate higher than or equal to R_2. The rate pair (R_1, R_2) is said to be s-achievable if (R_1, R_2) is achievable by some four-session scheme τ with configuration s.

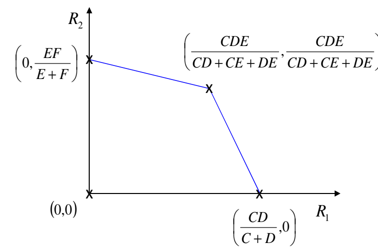


Fig. 2. $\mathcal{T}_4^*$.

Definition 4: The four-session achievable information rate region, denoted by $\mathcal{T}_4$, is the set

$$\left\{ (R_1, R_2) \in \mathbb{R}^2 \;\middle|\; \begin{array}{l} (R_1, R_2) \text{ is achievable by} \\ \text{some four-session scheme.} \end{array} \right\}.$$

Let $\mathcal{T}_4^*$ denote

$$\left\{ \begin{array}{l} (R_1, R_2) \\ \in \mathbb{R}^2 \end{array} \;\middle|\; \begin{array}{l} R_1 \geq 0, R_2 \geq 0, \\ R_2 \leq \frac{EF}{E+F} - R_1 \left(\frac{CEF+DEF-CDE}{CD(E+F)} \right), \\ R_2 \leq E - R_1 \left(\frac{CE+DE}{CD} \right). \end{array} \right\}, \tag{3}$$

which is shown in Figure 2, and $\mathcal{S}_4$ denote

$$\left\{ (R_1, R_2) \in \mathbb{R}^2 \;\middle|\; \begin{array}{c} R_1 \geq 0, R_2 \geq 0, \\ p_1 \geq 0, p_2 \geq 0, p_3 \geq 0, p_4 \geq 0, \\ p_1 + p_2 + p_3 + p_4 \leq 1, \\ R_1 \leq \min\{p_1 C, p_3 D\}, \\ R_2 \leq \min\{p_2 E, p_3 D + p_4 F\}. \end{array} \right\}. \tag{4}$$

We will show in this section that $\mathcal{T}_4 = \mathcal{S}_4 = \mathcal{T}_4^*$.

Lemma 1: Let $s = [\, p_1\ p_2\ p_3\ p_4 \,]$ be a four-session allocation. If (R_1, R_2) is s-achievable, then $R_1 \leq \min\{p_1 C, p_3 D\}$ and $R_2 \leq \min\{p_2 E, p_3 D + p_4 F\}$.

Proof: The lemma is intuitive and the proof is deferred to the appendix. ■

Lemma 2: Suppose $[\, p_1\ p_2\ p_3\ p_4 \,]$ is an allocation and (R_1, R_2) is a non-negative pair such that $R_1 \leq \min\{p_1 C, p_3 D\}$ and $R_2 \leq \min\{p_2 E, p_3 D + p_4 F\}$. Then, there exists an allocation $[\, p_1'\ p_2'\ p_3'\ p_4' \,]$ such that $R_1 = p_1' C = p_3' D$ and $R_2 \leq \min\{p_2' E, p_3' D + p_4' F\}$.

Proof: The lemma can be easily derived from Lemma 1 and the proof is deferred to the appendix. ■

Lemma 3: $\mathcal{S}_4 \subset \mathcal{T}_4^*$.

Proof: Since $\mathcal{S}_4$ is the projection of a 6-dimensional polyhedron on a 2-dimensional space, it can be shown by Fourier-Motzkin elimination [18] that $\mathcal{S}_4 = \mathcal{T}_4^*$. Since Fourier-Motzkin elimination is very tedious, we present a simpler alternative proof in the appendix. ■

Lemma 4: $\mathcal{T}_4 \subset \mathcal{S}_4 \subset \mathcal{T}_4^*$.

Proof: We have shown that $\mathcal{S}_4 \subset \mathcal{T}_4^*$ in Lemma 3. It remains to show that $\mathcal{T}_4 \subset \mathcal{S}_4$. Suppose (R_1, R_2) is in $\mathcal{T}_4$. Then, (R_1, R_2) is s-achievable for some allocation $s = [\, p_1\ p_2\ p_3\ p_4 \,]$. Using Lemma 1, we obtain $R_1 \leq$

τ_1				τ_2				τ_3			
p_1	0	p_1C	0	p_1	DE/(CD+CE+DE)	p_1C	CDE/(CD+CE+DE)	p_1	D/(C+D)	p_1C	CD/(C+D)
p_2	F/(E+F)	p_2E	EF/(E+F)	p_2	CD/(CD+CE+DE)	p_2E	CDE/(CD+CE+DE)	p_2	0	p_2E	0
p_3	0	p_3D	0	p_3	CE/(CD+CE+DE)	p_3D	CDE/(CD+CE+DE)	p_3	C/(C+D)	p_3D	CD/(C+D)
p_4	E/(E+F)	p_4F	EF/(E+F)	p_4	0	p_4F	0	p_4	0	p_4F	0

TABLE I
THE ALLOCATIONS OF τ_1, τ_2 AND τ_3 AND THE AVERAGE TRANSMISSION RATES FOR TYPE 1, 2, 3 AND 4 TRANSMISSIONS.

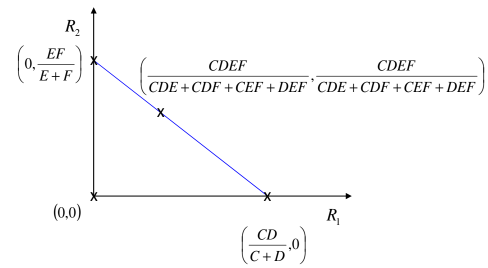


Fig. 3. The set of rate pairs achievable by four-session routing schemes.

$\min\{p_1C, p_3D\}$ and $R_2 \leq \min\{p_2E, p_3D + p_4F\}$, which implies that (R_1, R_2) is in $\mathcal{S}_4$ (cf. (4)). ■

Proposition 5: $\mathcal{T}_4^*$ is the convex hull of $(0,0)$, $(0, \frac{EF}{E+F})$, $(\frac{CDE}{CD+CE+DE}, \frac{CDE}{CD+CE+DE})$ and $(\frac{CD}{C+D}, 0)$.

Proof: The proposition can be easily verified and the proof is deferred to the appendix. ■

Lemma 6: $\mathcal{T}_4 \supset \mathcal{T}_4^*$. In particular, the four extreme points of $\mathcal{T}_4^*$, which are $(0,0)$, $(0, \frac{EF}{E+F})$, $(\frac{CDE}{CD+CE+DE}, \frac{CDE}{CD+CE+DE})$ and $(\frac{CD}{C+D}, 0)$, are achievable by some four-session schemes that involve linear operations only.

Proof: We only need to show that the four extreme points of $\mathcal{T}_4^*$ are achievable by some four-session network coding schemes. Then any other point in $\mathcal{T}_4^*$ can be achieved by time sharing of these schemes by Proposition 5. The rate pair $(0,0)$ is trivially achievable by any transmission scheme. We will propose three transmission schemes τ_1, τ_2 and τ_3 and show that they achieve the rate pairs $(0, \frac{EF}{E+F})$, $(\frac{CDE}{CD+CE+DE}, \frac{CDE}{CD+CE+DE})$ and $(\frac{CD}{C+D}, 0)$ respectively. To facilitate understanding, the allocations $[\, p_1\ p_2\ p_3\ p_4 \,]$ of τ_1, τ_2 and τ_3 and the average transmission rates p_1C, p_2E, p_3D and p_4F for Type 1, 2, 3 and 4 transmissions respectively are displayed in Table I followed by the descriptions of the schemes.

Under scheme τ_1, the average rates of the messages sent from node t_2 to r and from r to node t_1 are both $EF/(E+F)$. Node t_2 transmits its messages to r in the second session and r forwards the messages to node t_1 in the fourth session. Therefore, the rate pair $(0, \frac{EF}{E+F})$ is achievable by scheme τ_2.

Under scheme τ_2, the average rates of the messages sent from node t_1 to r, from node t_2 to r, from r to node t_1 and from r to node t_2 are all $CDE/(CD + CE + DE)$. Node t_1 transmits its messages to r in the first session and node t_2 transmits its messages to r in the second session. In the third session, relay r performs XOR operations between the messages from node t_1 and the messages from node t_2 bit by bit and broadcasts the resultant messages to both node t_1 and node t_2. In addition, node t_1 can recover the messages of node t_2 by performing XOR operations between its own messages and the messages from r bit by bit. Similarly, node t_2 can recover the messages of node t_1 by performing XOR operations between its own messages and the messages from r bit by bit. Consequently, the rate pair $(\frac{CDE}{CD+CE+DE}, \frac{CDE}{CD+CE+DE})$ is achievable by scheme τ_2.

Under scheme τ_3, the average rates of the messages sent from node t_1 to r and from r to node t_2 are both $CD/(C + D)$. Node t_1 transmits its messages to r in the first session and r forwards the messages to node t_2 in the third session. Therefore, the rate pair $(\frac{CD}{C+D}, 0)$ is achievable by scheme τ_3. ■

Theorem 1: $\mathcal{T}_4 = \mathcal{S}_4 = \mathcal{T}_4^*$.

Proof: It follows from Lemmas 4 and Lemma 6. ■

The achievable rate region $\mathcal{T}_4$ is the same as $\mathcal{T}_4^*$ by Theorem 1 and is shown in Figure 2. The extreme points $(0, \frac{EF}{E+F})$, $(\frac{CDE}{CD+CE+DE}, \frac{CDE}{CD+CE+DE})$ and $(\frac{CD}{C+D}, 0)$ of $\mathcal{T}_4$ correspond to the rate pairs achievable by schemes τ_1, τ_2 and τ_3 in Lemma 6 respectively. In addition, for any rate pair in $\mathcal{T}_4$, Lemma 6 provides a procedure to construct a network code that achieves the rate pair. The network code involves linear operations only since τ_1, τ_2 and τ_3 involve linear operations only. Consequently, the network code is practical due to its low complexity.

IV. Advantage of Network Coding

For any four-session routing scheme in the three-node point-to-point relay network, the relay receives the messages from node t_1 followed by forwarding the messages to node t_2, and similar procedures are performed in the opposite direction. Let p_1, p_2, p_3 and p_4 denote the fractions of time allocated to the transmission links (t_1, r), (t_2, r), (r, t_2) and (r, t_1) respectively. It is readily observed that the transmission rate from node t_1 to node t_2 and the transmission rate from node t_2 to node t_1 under the optimal routing scheme are $\min\{p_1C, p_3D\}$ and $\min\{p_2E, p_4F\}$ respectively. Therefore, the set of rate pairs achievable by the four-session routing schemes on the three-node point-to-point relay network is

$$\left\{ (R_1, R_2) \in \mathbb{R}^2 \;\middle|\; \begin{array}{c} R_1 \geq 0, R_2 \geq 0, \\ p_1 \geq 0, p_2 \geq 0, p_3 \geq 0, p_4 \geq 0, \\ p_1 + p_2 + p_3 + p_4 \leq 1, \\ R_1 \leq \min\{p_1C, p_3D\}, \\ R_2 \leq \min\{p_2E, p_4F\}. \end{array} \right\}. \quad (5)$$

Following similar procedures for proving that (3) and (4) are equal in Section III, we can obtain that (5) is equal to

$$\left\{ (R_1, R_2) \in \mathbb{R}^2 \;\middle|\; \begin{array}{c} R_1 \geq 0, R_2 \geq 0, \\ R_2 \leq \frac{EF}{E+F} - R_1 \left(\frac{EF(C+D)}{CD(E+F)}\right). \end{array} \right\}, \quad (6)$$

which is shown in Figure 3.

The rate region achievable by four-session schemes was obtained in the previous section and it is shown in Figure 2. It is readily observed by comparing Figure 3 with Figure 2 that the use of network coding always enlarges the achievable rate region. Using the fact that the maximum equal-rate pair achievable by four-session routing schemes lies on the straight line connecting the points $(\frac{CD}{C+D}, 0)$ and $(0, \frac{EF}{E+F})$, we obtain that the maximum equal-rate pair achievable by four-session routing schemes is $\frac{CDEF}{CDE+CDF+CEF+DEF}$. Since the maximum equal-rate pair in $\mathcal{T}_4$ is $\frac{CDE}{CD+CE+DE}$ (cf. Figure 2) and $\frac{CDE}{CD+CE+DE} - \frac{CDEF}{CDE+CDF+CEF+DEF} > 0$, the use of network coding rather than routing alone increases the maximum equal-rate throughput in the three-node point-to-point relay network. Since the maximum equal-rate pair in $\mathcal{T}_4$ is upper bounded by $\min\{\frac{CD}{C+D}, \frac{EF}{E+F}\}$ (cf. Figure 2), the fractional increase in the maximum equal-rate throughput is upper bounded by

$$\begin{aligned}
&\frac{\min\{\frac{CD}{C+D}, \frac{EF}{E+F}\} - \frac{CDEF}{CDE+CDF+CEF+DEF}}{\frac{CDEF}{CDE+CDF+CEF+DEF}} \\
&= \min\left\{\frac{CD(E+F)}{EF(C+D)}, \frac{EF(C+D)}{CD(E+F)}\right\} \\
&\leq 1.
\end{aligned}$$

V. Conclusion

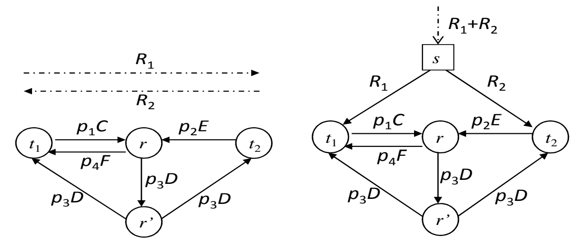
An achievable rate region for the three-node point-to-point relay network is obtained. A procedure for constructing network codes that achieve the rate pairs in the achievable rate region is presented. Since only linear operations are required for the network codes, the network codes are practical due to their low complexity. In addition, we show that the use of network coding rather than routing alone enlarges the achievable rate region, in particular increases the maximum equal-rate throughput.

Appendix

Proof of Lemma 1: Suppose (R_1, R_2) is s-achievable. Under any four-session transmission scheme τ with configuration $s = [\ p_1\ p_2\ p_3\ p_4\ ]$,

1) the highest average rate that node t_1 can transmit messages reliably to r in the first session is $p_1 C$;
2) the highest average rate that node t_2 can transmit messages reliably to r in the second session is $p_2 E$;
3) the highest average rate that r can broadcast messages reliably to both node t_1 and node t_2 in the third session is $p_3 D$;
4) the highest average rate that r can transmit messages reliably to node t_1 in the fourth session is $p_4 F$.

Therefore, (R_1, R_2) must satisfy the rate constraints induced by the two-source point-to-point network in Figure 4(a), where the introduction of the dummy node r' captures the broadcast nature of Type 3 transmissions. We refer the reader to [16, p.415] for the use of the dummy broadcast node. Since node t_1 and node t_2 exchange independent messages in such a way



(a) A two-source point-to-point network.

(b) A single-source point-to-point multicast network.

Fig. 4. Two equivalent network coding problems.

that they eventually possess the same set of messages after the information exchange, the two-source network coding problem for the network in Figure 4(a) is equivalent to the single-source multicast problem for the network in Figure 4(b), where node t_1 and node t_2 are the receivers and s is the source node of the network. We refer the reader to [16, p.459] for the equivalence of the two problems. Applying the cut-set bound in [16, p.429] for four different cuts of the network in Figure 4(b), we obtain

$$\begin{aligned}
R_1 + R_2 &\leq R_2 + p_1 C, \\
R_1 + R_2 &\leq R_2 + p_3 D, \\
R_1 + R_2 &\leq R_1 + p_2 E
\end{aligned}$$

and

$$R_1 + R_2 \leq R_1 + p_3 D + p_4 F.$$

Therefore, $R_1 \leq p_1 C$, $R_1 \leq p_3 D$, $R_2 \leq p_2 E$ and $R_2 \leq p_3 D + p_4 F$, and the lemma follows. ■

Proof of Lemma 2: There exist $p_1' \leq p_1$ and $p_3' \leq p_3$ such that $R_1 = p_1' C = p_3' D$. Let $p_2' = p_2$ and $p_4' = p_4 + p_3 - p_3'$. Then,

$$\begin{aligned}
R_2 &\leq \min\{p_2 E, p_3 D + p_4 F\} \\
&\leq \min\{p_2' E, p_3' D + (p_3 - p_3')F + p_4 F\} \\
&= \min\{p_2' E, p_3' D + p_4' F\},
\end{aligned}$$

where the second inequality follows from (1). Since

$$p_1' + p_2' + p_3' + p_4' \leq p_1 + p_2 + p_3' + (p_4 + p_3 - p_3') \leq 1,$$

it follows that $[\ p_1'\ p_2'\ p_3'\ p_4'\ ]$ is the desired allocation. ■

Proof of Lemma 3: Suppose (R_1, R_2) is in $\mathcal{S}_4$. Then, (R_1, R_2) is a non-negative pair such that $R_1 \leq \min\{p_1 C, p_3 D\}$ and $R_2 \leq \min\{p_2 E, p_3 D + p_4 F\}$ for some allocation $[\ p_1\ p_2\ p_3\ p_4\ ]$. By Lemma 2, there exists an allocation $[\ p_1'\ p_2'\ p_3'\ p_4'\ ]$ such that

$$R_1 = p_1' C = p_3' D \tag{7}$$

and

$$R_2 \leq \min\{p_2' E, p_3' D + p_4' F\}. \tag{8}$$

We will show that (R_1, R_2) satisfies the four inequalities defining $\mathcal{T}_4^*$ (cf. (3)), which will imply that $(R_1, R_2) \in \mathcal{T}_4^*$. The lemma then follows. Clearly, $R_1 \geq 0$ and $R_2 \geq 0$. Consider the following chain of inequalities:

$$\begin{aligned}
&R_2 - \frac{EF}{E+F} + R_1\left(\frac{CEF+DEF-CDE}{CD(E+F)}\right) \\
&= \frac{1}{CD(E+F)}(CDER_2 + CDFR_2 - CDEF + \\
&\quad CEFR_1 + DEFR_1 - CDER_1) \\
&\overset{(a)}{\leq} \frac{1}{CD(E+F)}(CDEFp_4' + CD^2Ep_3' + CDEFp_2' - \\
&\quad CDEF + CDEFp_3' + CDEFp_1' - CD^2Ep_3') \\
&= \frac{EF}{E+F}(p_1' + p_2' + p_3' + p_4' - 1) \\
&\overset{(b)}{\leq} 0,
\end{aligned}$$

where (a) follows from (7) and (8), and (b) follows from (2). Therefore, $R_2 \leq \frac{EF}{E+F} - R_1\left(\frac{CEF+DEF-CDE}{CD(E+F)}\right)$.
Consider the following chain of inequalities:

$$\begin{aligned}
&R_2 - E + R_1\left(\frac{CE+DE}{CD}\right) \\
&= \frac{CDR_2 - CDE + CER_1 + DER_1}{CD} \\
&\overset{(a)}{\leq} \frac{CDEp_2' - CDE + CDEp_3' + CDEp_1'}{CD} \\
&= E(p_1' + p_2' + p_3' - 1) \\
&\overset{(b)}{\leq} -p_4E \\
&\leq 0,
\end{aligned}$$

where (a) follows from (7) and (8), and (b) follows from (2). Therefore, $R_2 \leq E - R_1\left(\frac{CE+DE}{CD}\right)$. ∎

Proof of Proposition 5: We will show that $(0,0)$, $(0, \frac{EF}{E+F})$, $(\frac{CDE}{CD+CE+DE}, \frac{CDE}{CD+CE+DE})$ and $(\frac{CD}{C+D}, 0)$ are all the vertices of $\mathcal{T}_4^*$. The proposition then follows. Let $L_1 : R_1 = 0$, $L_2 : R_2 = 0$, $L_3 : R_2 = \frac{EF}{E+F} - R_1\left(\frac{CEF+DEF-CDE}{CD(E+F)}\right)$ and $L_4 : R_2 = E - R_1(\frac{CE+DE}{CD})$ be the four lines defining the boundary of $\mathcal{T}_4^*$ (cf. (3)). The x-intercept and y-intercept of L_3 are $\frac{CDF}{CF+DF-CD}$ and $\frac{EF}{E+F}$ respectively. The x-intercept and y-intercept of L_4 are $\frac{CD}{C+D}$ and E respectively. It can be verified that $\frac{CDF}{CF+DF-CD} - \frac{CD}{C+D} > 0$ (cf. (1)) and $E - \frac{EF}{E+F} > 0$, which implies that the x-intercept of L_3 is larger than that of L_4 and the y-intercept of L_4 is larger than that of L_3. By inspecting Figure 5, we observe that the vertices of $\mathcal{T}_4^*$ are $(0,0)$, $(0, \frac{EF}{E+F})$, $(\frac{CD}{C+D}, 0)$ and $(\frac{CDE}{CD+CE+DE}, \frac{CDE}{CD+CE+DE})$, the intersection point between L_3 and L_4. ∎

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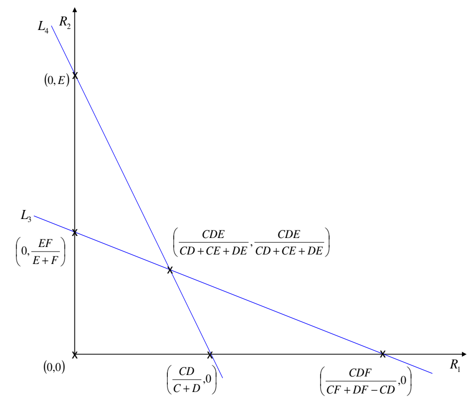

Fig. 5. L_3 and L_4.

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